Quantum Block and Convolutional Codes from Self-orthogonal Product Codes

Markus Grassl
Institut für Algorithmen und Kognitive Systeme
Arbeitsgruppe Quantum Computing
Fakultät für Informatik, Universität Karlsruhe (TH)
Am Fasanengarten 5, 76 128 Karlsruhe, Germany
Email: grassl@ira.uka.de

Martin Rötteler
NEC Labs America, Inc.
4 Independence Way
Princeton, NJ 08540, USA
Email: mroetteler@nec-labs.com

***Abstract*— We present a construction of self-orthogonal codes using product codes. From the resulting codes, one can construct both block quantum error-correcting codes and quantum convolutional codes. We show that from the examples of convolutional codes found, we can derive ordinary quantum error-correcting codes using tail-biting with parameters $[\![42N, 24N, 3]\!]_2$. While it is known that the product construction cannot improve the rate in the classical case, we show that this can happen for quantum codes: we show that a code $[\![15, 7, 3]\!]_2$ is obtained by the product of a code $[\![5, 1, 3]\!]_2$ with a suitable code.**

I. Introduction

Quantum convolutional codes are motivated by their classical counterparts [3]. As in the classical case the idea is to allow for the protection of arbitrary long streams of information in such a way that as many errors as possible can be corrected. To achieve this the information is "smeared out" to the output stream by adding a certain amount of redundancy, but at the same time meeting the requirement to be local, *i. e.*, encoding/decoding can be done by a processes which needs only a constant amount of memory. In [13] the basic theory of quantum convolutional codes has been developed. There it has been shown that, similar to the classical codes, quantum convolutional codes can be decoded by a maximum likelihood error estimation algorithm which has linear complexity. However, the authors only gave an example of one (rate $1/5$) quantum convolutional code. This research was motivated by the question to find new examples of quantum convolutional codes. The construction presented in this paper resorts on the idea of product codes. An extra requirement imposed by the applicability to quantum codes is that the dual distance has to be high. The main source of the examples presented at the end of the paper are two-dimensional cyclic codes (sometimes also called "bicyclic codes"). We apply this to the situation where the code is a product code of two Reed-Solomon codes.

II. Self-orthogonal Product Codes

A. Quantum error-correcting codes from classical codes

Most of the constructions for quantum error-correcting codes (QECCs) for a quantum system of dimension q (*qudits*), where $q = p^\ell$ is a prime power, are based on classical error-correcting codes over $GF(q)$ or $GF(q^2)$. The so-called CSS codes (see [5], [14]) are based on linear codes C_1 and C_2 over $GF(q)$ with $C_2^\perp \subseteq C_1$. Here $C_2^\perp$ is the dual code of C_2 with respect to the Euclidean inner product. In particular, if $C = C_1 = C_2$ this implies that $C^\perp$ is a weakly self-dual code. The construction can be summarized as follows:

Lemma 1: Let $C = [n, k, d]_q$ be a weakly self-dual linear code, *i. e.*, $C \subseteq C^\perp = [n, n-k, d^\perp]_q$. Then a quantum error-correcting code encoding $n-2k$ qudits using n qudits, denoted by $\mathcal{C} = [\![n, n-2k, d_q \geq d^\perp]\!]_q$ exists.

Another class of quantum codes can be obtained from codes over $GF(q^2)$ which are self-orthogonal with respect to the Hermitian inner product, denoted by $C \subseteq C^*$. Both cases can be generalized to a construction of QECCs based on additive codes over $GF(q^2)$ which are self-orthogonal with respect to the symplectic (trace) inner product, *i. e.* $C \subseteq C^\star$ [1].

B. Inner products on vector spaces over $GF(q)$ and $GF(q^2)$

In this paper, we will use three different inner products on vector spaces over $GF(q)$ and $GF(q^2)$ which are defined as follows:

Euclidean:

$$\boldsymbol{v} \cdot \boldsymbol{w} := \sum_{i=1}^n v_i w_i \quad \text{for } \boldsymbol{v}, \boldsymbol{w} \in GF(q)^n \tag{1}$$

Hermitian:

$$\boldsymbol{v} * \boldsymbol{w} := \sum_{i=1}^n v_i w_i^q \quad \text{for } \boldsymbol{v}, \boldsymbol{w} \in GF(q^2)^n \tag{2}$$

symplectic:

$$\boldsymbol{v} \star \boldsymbol{w} := \sum_{i=1}^n \operatorname{tr}(v_i w_i^q) \quad \text{for } \boldsymbol{v}, \boldsymbol{w} \in GF(q^2)^n, \tag{3}$$

where $\operatorname{tr}(x)$ denotes the trace of $GF(q^2)$ over its prime field $GF(p)$. Both the Euclidean and the Hermitian inner product are bilinear over $GF(q)$ respectively $GF(q^2)$, but the symplectic inner product is only $GF(p)$-bilinear because of the trace map. For codes which are linear over $GF(q)$, linear over $GF(p^2)$, or additive (*i. e.* $GF(p)$-linear), one can define a dual code with respect to the inner products (1), (2), or (3), respectively. The three cases are summarized in Table I.

Next, we consider inner products on tensor products of vector spaces.

TABLE I

NOTATION USED FOR THE THREE DIFFERENT INNER PRODUCTS AND THE CORRESPONDING DUAL CODES.

	dual code	inner product	linear over
Euclidean	$C^\perp$	$\boldsymbol{v}\cdot\boldsymbol{w}$	$GF(q)$
Hermitian	C^*	$\boldsymbol{v}*\boldsymbol{w}$	$GF(q^2)$
symplectic	$C^\star$	$\boldsymbol{v}\star\boldsymbol{w}$	$GF(p)$

Lemma 2: For all $\boldsymbol{v},\boldsymbol{v}'\in GF(q)^n$ and $\boldsymbol{w},\boldsymbol{w}'\in GF(q)^m$, we have

$$(\boldsymbol{v}\otimes\boldsymbol{w})\cdot(\boldsymbol{v}'\otimes\boldsymbol{w}')=(\boldsymbol{v}\cdot\boldsymbol{v}')(\boldsymbol{w}\cdot\boldsymbol{w}'),\tag{4}$$

i. e., the Euclidean inner product is compatible with the tensor product of vector spaces over $GF(q)$. Furthermore, for all $\boldsymbol{v},\boldsymbol{v}'\in GF(q^2)^n$ and $\boldsymbol{w},\boldsymbol{w}'\in GF(q^2)^m$, we have

$$(\boldsymbol{v}\otimes\boldsymbol{w})*(\boldsymbol{v}'\otimes\boldsymbol{w}')=(\boldsymbol{v}*\boldsymbol{v}')(\boldsymbol{w}*\boldsymbol{w}'),\tag{5}$$

i. e., the Hermitian inner product is compatible with the tensor product of vector spaces over $GF(q^2)$.

Proof: The tensor product of two vectors is given by $(\boldsymbol{v}\otimes\boldsymbol{w})=(v_iw_j)_{i,j}$. Then for the Euclidean inner product we get

$$\begin{aligned}&(\boldsymbol{v}\otimes\boldsymbol{w})\cdot(\boldsymbol{v}'\otimes\boldsymbol{w}')\\&\quad=\sum_{i=1}^{n}\sum_{j=1}^{m}v_iw_jv_i'w_j'=\Bigl(\sum_{i=1}^{n}v_iv_i'\Bigr)\Bigl(\sum_{j=1}^{m}w_jw_j'\Bigr)\\&\quad=(\boldsymbol{v}\cdot\boldsymbol{v}')(\boldsymbol{w}\cdot\boldsymbol{w}').\end{aligned}$$

Similarly, for the Hermitian inner product we get

$$\begin{aligned}&(\boldsymbol{v}\otimes\boldsymbol{w})*(\boldsymbol{v}'\otimes\boldsymbol{w}')\\&\quad=\sum_{i=1}^{n}\sum_{j=1}^{m}v_iw_j(v_i'w_j')^q=\Bigl(\sum_{i=1}^{n}v_i{v_i'}^q\Bigr)\Bigl(\sum_{j=1}^{m}w_j{w_j'}^q\Bigr)\\&\quad=(\boldsymbol{v}*\boldsymbol{v}')(\boldsymbol{w}*\boldsymbol{w}').\end{aligned}$$

■

For the symplectic inner product, the situation is a bit more complicated as it is only $GF(p)$-linear. Considering $GF(q)^m$ only as vector space over $GF(p)$, we may define the $GF(p)$ tensor product of $V_1=GF(p)^n$ and $V_2=GF(q)^m$, denoted by $V_1\otimes_p V_2$.

Lemma 3: For all $\boldsymbol{v},\boldsymbol{v}'\in GF(p)^n$ and $\boldsymbol{w},\boldsymbol{w}'\in GF(q)^m$, we have $(\boldsymbol{v}\otimes_p\boldsymbol{w})\star(\boldsymbol{v}'\otimes_p\boldsymbol{w}')=(\boldsymbol{v}\cdot\boldsymbol{v}')(\boldsymbol{w}\star\boldsymbol{w}')$, *i. e.*, the symplectic inner product on the $GF(p)$ tensor product space is the product of the Euclidean inner product on the first space and the symplectic inner product on the second.

Proof: Similar to the proof of Lemma 2, we compute

$$\begin{aligned}(\boldsymbol{v}\otimes\boldsymbol{w})\star(\boldsymbol{v}'\otimes\boldsymbol{w}')&=\sum_{i=1}^{n}\sum_{j=1}^{m}\operatorname{tr}\left(v_iw_j(v_i'w_j')^q\right)\\&=\operatorname{tr}\left(\Bigl(\sum_{i=1}^{n}v_i{v_i'}^q\Bigr)\Bigl(\sum_{j=1}^{m}w_j{w_j'}^q\Bigr)\right).\end{aligned}$$

As $\boldsymbol{v}$ and $\boldsymbol{v}'$ are vectors over the prime field, the left factor equals their Euclidean inner product $\boldsymbol{v}\cdot\boldsymbol{v}'$ which takes values in $GF(p)$ only. Using the $GF(p)$-linearity of the trace map, the proof is completed. ■

C. Product codes

Next we present the fundamental properties of the product of two codes which combines two codes (see *e. g.* [2], [11]).

Lemma 4: Let $C_1=[n_1,k_1,d_1]_q$ and $C_2=[n_2,k_2,d_2]_q$ be linear codes over $GF(q)$ with generator matrices $G^{(1)}$ and $G^{(2)}$, respectively. Then the product code $C_\pi:=C_1\otimes C_2$ is a linear code $C_\pi:=[n_1n_2,k_1k_2,d_1d_2]_q$ generated by the matrix $G:=G^{(1)}\otimes G^{(2)}$, where $\otimes$ denotes the Kronecker product, *i. e.*

$$G:=\begin{pmatrix}g^{(1)}_{11}G^{(2)} & g^{(1)}_{12}G^{(2)} & \dots & g^{(1)}_{1,n_1}G^{(2)}\\ g^{(1)}_{21}G^{(2)} & g^{(1)}_{22}G^{(2)} & \dots & g^{(1)}_{2,n_1}G^{(2)}\\ \vdots & \vdots & \ddots & \vdots\\ g^{(1)}_{k_1,1}G^{(2)} & g^{(1)}_{k_1,2}G^{(2)} & \dots & g^{(1)}_{k_1,n_1}G^{(2)}\end{pmatrix}.\tag{6}$$

If $C_1=[n_1,k_1,d_1]_p$ is a linear code over the prime field $GF(p)$ and $C_2=(n_2,p^{k_2},d_2)_q$ is an additive code over $GF(q)$, then $C_{\pi,p}:=C_1\otimes_p C_2$ is an additive code with parameters $C_{\pi,p}=(n_1n_2,p^{k_1k_2},d_1d_2)_q$.

The following theorem is valid for all compatible choices of inner products on the component spaces of a tensor product space and the tensor product space itself.

Theorem 5: Let $C_\pi=C_1\otimes C_2$ be the product code of the codes $C_1=[n_1,k_1,d_1]$ and $C_2=[n_2,k_2,d_2]$. By H_1 and H_2 we denote generator matrices of the corresponding dual codes. Furthermore, let A_1 and A_2 be matrices of size $k_1\times n_1$ and $k_2\times n_2$, respectively, such that the row span of the matrices H_1 and A_1 is the full vector space and similar for H_2 and A_2. Then a generator matrix H of the dual code of C_π is given by

$$H:=\begin{pmatrix}H_1\otimes H_2\\A_1\otimes H_2\\H_1\otimes A_2\end{pmatrix}.\tag{7}$$

Proof: Let V_1 and V_2 be the full vector spaces containing the codes C_1 and C_2. Furthermore, by D_1 and D_2 we denote the dual code of C_1 and C_2 with respect to the inner product on V_1 and V_2, respectively. Using the properties of the inner products on tensor product spaces (see Lemma 2 and Lemma 3), it is obvious that the dual code D_π of C_π contains both $V_1\otimes D_2$ and $D_1\otimes V_2$. The intersection of these spaces is $D_0:=D_1\otimes D_2$, spanned by $H_1\otimes H_2$. The complement of D_0 in $V_1\otimes D_2$ is spanned by $A_1\otimes H_2$, and analogously for the complement of D_0 in $D_1\otimes V_2$. Hence D_π can be decomposed as

$$D_\pi=\Bigl(D_1\otimes D_2\Bigr)\oplus\Bigl(\langle A_1\rangle\otimes D_2\Bigr)\oplus\Bigl(D_1\otimes\langle A_2\rangle\Bigr).$$

Here $\langle A\rangle$ denotes the row span of the matrix A. Considering the dimension of the spaces, the result follows. ■

Corollary 6: The minimum distance of the dual of the product code $C_\pi = C_1 \otimes C_2$ cannot exceed the minimum of the dual distance of C_1 and the dual distance of C_2.

Proof: The dual code D_π of C_π contains $V_1 \otimes D_2$, *i. e.*, the product of the trivial code $[n_1, n_1, 1]$ and D_2. Hence the minimum distance of D_π cannot be larger than that of D_2. The result follows by interchanging the role of C_1 and C_2. ■

Note that despite their poor behavior in terms of minimum distance, the dual of product codes can be used for burst error correction (see [6], [15]). For the construction of QECCs, we will make use of the following property.

Theorem 7: Let $C_E \subseteq C_E^\perp$, $C_H \subseteq C_H^*$, and $C_s \subseteq C_s^\star$ denote codes which are self-orthogonal with respect to the inner products (1), (2), or (3), respectively. Furthermore, let C denote an arbitrary linear code over $GF(q)$, respectively $GF(q^2)$, and let C_p be a linear code over $GF(p)$. Then

(i) $C \otimes C_E$ is Euclidean self-orthogonal.
(ii) $C \otimes C_H$ is Hermitian self-orthogonal.
(iii) $C_p \otimes_p C_s$ is symplectic self-orthogonal.

Proof: The result directly follows using Lemma 2, Lemma 3, and Theorem 5. ■

III. Product Codes from Cyclic Codes

In this section we investigate the product of two cyclic codes (see [2, Chapter 10.4], [3, Chapter 10.2]).

Let $C_1 = [n_1, k_1]$ and $C_2 = [n_2, k_2]$ be cyclic linear codes with generator polynomials $g_1(X)$ and $g_2(Y)$. Then $C_\pi = C_1 \otimes C_2$ is a bicyclic code generated by $g_1(X)g_2(Y)$. The codewords of C_π correspond to all bivariate polynomials $c(X,Y) = i(X,Y)g_1(X)g_2(Y)$ modulo the ideal generated by $X^{n_1}-1$ and $Y^{n_2}-1$, where $i(X,Y) \in GF(q)[X,Y]$ is an arbitrary bivariate polynomial. The two-dimensional spectrum of $c(X,Y)$ is the $n_1 \times n_2$ matrix $(\hat{c}_{i,j})$ with entries

$$\hat{c}_{i,j} := c(\alpha^i, \beta^j), \tag{8}$$

where α and β are primitive roots of unity of order n_1 and n_2, respectively. The spectrum $\hat{c}$ is zero in all vertical stripes corresponding to the roots α^i of $g_1(X)$ and in all horizontal stripes corresponding to the roots β^j of $g_2(X)$ (see Fig. 1 a)). The generator polynomial $h_1(X)$ of the Euclidean dual $C_1^\perp$ is the reciprocal polynomial of $(X^{n_1}-1)/g_1(X)$. Hence its one-dimensional spectrum is zero at the negative of those positions where the spectrum of the code C_1 takes arbitrary values (*cf.* Fig. 2). For the generator polynomial $h_2(Y)$ of $C_2^\perp$ the analogous statement is true. Therefore the Euclidean dual code $(C_1 \otimes C_2)^\perp$ of the product code $C_1 \otimes C_2$ consists of all polynomials that are multiples of $h_1(X)$ or $h_2(Y)$. Interchanging the zeros and blanks in the two-dimensional spectrum of the product code and applying the coordinate map (*cf.* Fig. 2) to both the rows and columns, we obtain the two-dimensional spectrum of the dual code $(C_1 \otimes C_2)^\perp$.

For the Hermitian dual code, we get analogous results. As the Hermitian inner product involves the Frobenius map $x \mapsto x^q$, the transformation on the coordinates now reads $i \mapsto -qi \bmod n_j$.

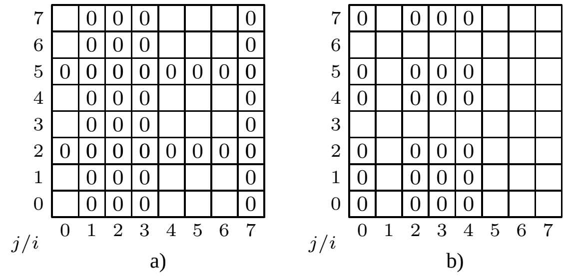


Fig. 1. Two-dimensional spectrum of a) the product of two cyclic codes and b) the dual code. Blank entries may take arbitrary value.

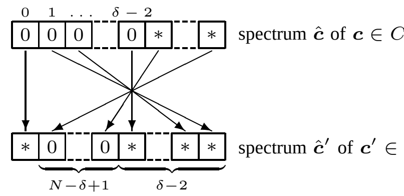


Fig. 2. Relation between the spectra of a Reed-Solomon code C and its dual. Positions taking arbitrary values (marked with $*$) and positions being zero are interchanged using to the map $i \mapsto -i \bmod (q-1)$ [9].

For Reed-Solomon codes, the picture simplifies. The two-dimensional spectrum of the product of two Reed-Solomon codes with minimum distance δ_1 and δ_2 corresponds to a vertical stripe of zeros of width $\delta_1 - 1$ and a horizontal stripe of height $\delta_2 - 1$. Without loss of generality, the stripes can be shifted such that the rectangle of arbitrary values is in the upper right corner (see Fig. 3 a). Then for the dual code, the spectrum is zero in a rectangle (see Fig. 3 b) whose width and height is determined by the dual distances $(q-\delta_1)$ and $(q-\delta_1)$. Using the BCH-like lower bound for bicyclic codes (see [3, p. 320]), we conclude that the minimum distance of the dual of the product code is $\min(q-\delta_1, q-\delta_2)$. In summary, we get the following theorem:

Theorem 8: The product code of two Reed-Solomon codes $C_1 = [q-1, q-\delta_1, \delta_1]_q$ and $C_2 = [q-1, q-\delta_2, \delta_2]_q$ over $GF(q)$ is

$$C_1 \otimes C_2 = [(q-1)^2, (q-\delta_1)(q-\delta_2), \delta_1\delta_2]_q. \tag{9}$$

The Euclidean dual code $(C_1 \otimes C_2)^\perp = [(q-1)^2, K^\perp, d^\perp]_q$

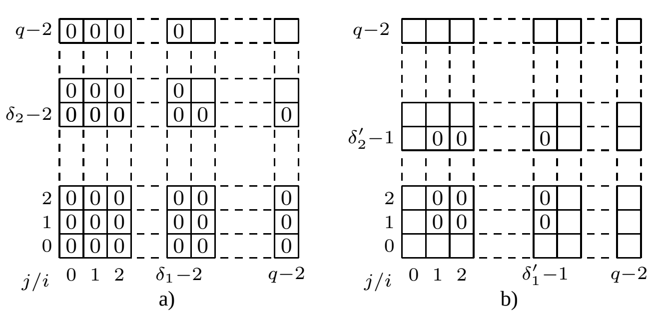


Fig. 3. Two-dimensional spectrum of a) the product of two Reed-Solomon codes C_1 and C_2 with minimum distance δ_1 and δ_2, and b) the dual code, where δ_1' and δ_2' denote minimum distance of the dual codes $C_1^\perp$ and $C_2^\perp$.

has parameters

$$K^{\perp} = q(d_1 + d_2 - 2) - d_1 d_2 + 1$$
$$d^{\perp} = \min(q - \delta_1, q - \delta_2).$$

Moreover, the product code is self-orthogonal if C_1 or C_2 is self-orthogonal.

Note that the result is still true when replacing the Reed-Solomon code over $GF(q)$ of length $(q-1)$ by a cyclic code $C = [n, k, d]_q$ with generator polynomial $g(X) = \prod_{i=0}^{d-2}(X - \alpha^i)$ where n is a divisor of $q-1$ and α is a primitive n-th root of unity.

IV. Quantum Codes from Product Codes

A. Quantum Block Codes

In the previous section we have seen that the product of a self-orthogonal Reed-Solomon code with an arbitrary Reed-Solomon codes yields a self-orthogonal product code. Using Lemma 1, we can construct quantum error-correcting codes.

Theorem 9: Let $C_1 = [q-1, \mu_1, q-\mu_1]_q$ and $C_2 = [q-1, \mu_2, q-\mu_2]_q$ be Reed-Solomon codes where $\mu_1 < (q-1)/2$. Then a quantum error-correcting code

$$\mathcal{C} = [\![(q-1)^2, (q-1)^2 - 2\mu_1\mu_2, 1 + \min(\mu_1, \mu_2)]\!]_q \quad (10)$$

exists.

Proof: For $\mu_1 < (q-1)/2$, the code C_1 is Euclidean self-orthogonal [10]. The dual distance of C_1 and C_2 is $\mu_1 + 1$ and $\mu_2 + 1$, respectively. By Theorem 8, the product code $C_\pi = C_1 \otimes C_2 = [(q-1)^2, \mu_1\mu_2, (q-\mu_1)(q-\mu_2)]_q$ is self-orthogonal. Its Euclidean dual has parameters $C_\pi^\perp = [(q-1)^2, (q-1)^2 - \mu_1\mu_2, 1 + \min(\mu_1, \mu_2)]_q$. Hence by Lemma 1 a QECC with the parameters given in eq. (10) exists. ■

Note that from C_1 and C_2 (provided $\mu_2 < (q-1)/2$), one can construct optimal QECCs with parameters $[\![q-1, q-2\mu-1, \mu+1]\!]_q$ (see [10]). The product of the rates of these codes is

$$\left(1 - \frac{2\mu_1}{q-1}\right)\left(1 - \frac{2\mu_2}{q-1}\right) = 1 - \frac{2(\mu_1+\mu_2)}{q-1} + \frac{4\mu_1\mu_2}{(q-1)^2}$$

The rate of the code of Theorem 9 is

$$1 - \frac{2\mu_1\mu_2}{(q-1)^2}.$$

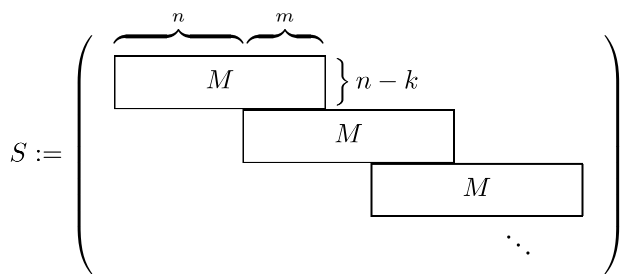
If we choose $\mu_1 = \mu_2$, we will obtain a QECC of squared length and the same minimum distance, but higher rate provided $\mu_1 = \mu_2 < 2(q-1)/3$.

Note that we can obtain good QECCs by this construction using other codes than Reed-Solomon codes. Let $C = [5, 2, 4]_4$ be the Hermitian dual of the quaternary Hamming code. Using $C \subseteq C^* = [5, 3, 3]_4$, an optimal QECC $\mathcal{C} = [\![5, 1, 3]\!]_2$ can be constructed. The code C is not a Reed-Solomon code, but its spectrum fulfills the conditions for Theorem 8. Hence the product of C with itself is a Hermitian self-orthogonal code $C \otimes C = [25, 4, 16]_4 \subseteq (C \otimes C)^* = [25, 21, 3]_4$. This yields a QECC $\mathcal{C}^{(2)} = [\![25, 17, 3]\!]_2$, whose rate is more than three times higher than that of $\mathcal{C}$.

The product code of C, considered as additive code, with the binary simplex code $C_1 = [3, 2, 2]_2$ is an additive code $C_2 := C_1 \otimes_p C = (15, 2^8, 8)_2$ which is contained in its symplectic dual $C^\star = (15, 2^{22}, 3)_2$. Hence we obtain a QECC $\mathcal{C}_\pi = [\![15, 7, 3]\!]_2$.

V. Quantum Convolutional Codes

Following [13], an (n, k, m) quantum convolutional code can be described in terms of a semi-infinite stabilizer matrix S. The matrix S has a block band structure where each block M has size $(n-k) \times (n+m)$. All blocks are equal. In the second block, the matrix M is shifted by n columns, hence any two consecutive blocks overlap in m positions. The general structure of the matrix is as follows:



The classical convolutional code generated by S must be self-orthogonal with respect to some of the inner products of Section II. The quantum product codes constructed in the previous section naturally lend themselves to convolutional codes because of the following observation. Let $M = G^{(1)} \otimes G^{(2)}$ be the generator matrix of $C_1 \otimes C_2$ as in eq. (6). Assume that $m = tn_2$ is a multiple of n_2, the length of C_2. Since C_2 is self-orthogonal, we have that the submatrix of M which consists of the last m columns of M is orthogonal to the submatrix which consists of the first m columns of M. Hence, we obtain a semi-infinite stabilizer matrix S by iterative shifting of the block M by $n_1 n_2 - m = (n_1 - t)n_2$ positions.

To give an example, we let $C = [7, 3, 4]_2$ be the Euclidean dual of the binary Hamming code. Using $C \subseteq C^\perp = [7, 4, 3]_2$, a QECC $\mathcal{C} = [\![7, 1, 3]\!]_2$ can be constructed. The product code of C with itself is a code $C_\pi = C \otimes C = [49, 9, 16]_2$ which is contained in its dual $C_\pi^\perp = [49, 40, 3]_2$. Hence we obtain a QECC $\mathcal{C}_\pi = [\![49, 31, 3]\!]_2$. The possible parameters for quantum convolutional codes obtained from the product code C_π by the CSS construction (*i. e.*, by considering the generator matrix $C_\pi \otimes GF(4)$) are $(49 - m, 31, m)$, $m = 7, 14$. The free distance of these codes is 3. Using tail-biting with $N \geq 2$ blocks and $m = 7$ (see [8]) we obtain QECCs $[\![42N, 24N, d]\!]_2$. Using Magma [4] we compute $d = 3$.

From the product code $\mathcal{C}_\pi = [\![15, 7, 3]\!]_2$ described above we can obtain a quantum convolutional code with parameters $(10, 7, 5)$, *i. e.*, we choose $m = 5$.

If the matrix M defining the semi-infinite band matrix S is the generator matrix $G^{(1)} \otimes G^{(2)}$ of a product code, the matrix S itself can be decomposed as a tensor product $S = S^{(1)} \otimes G^{(2)}$, provided the overlap m is a multiple of the length n_2 of the second code, *i. e.*, $m = tn_2$ (see Fig. 4). The matrix

$$\begin{pmatrix}
\begin{matrix} g_{11}^{(1)}G^{(2)} & g_{12}^{(1)}G^{(2)} & \cdots & g_{1,n_1}^{(1)}G^{(2)} \\ g_{21}^{(1)}G^{(2)} & g_{22}^{(1)}G^{(2)} & \cdots & g_{2,n_1}^{(1)}G^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k_1,1}^{(1)}G^{(2)} & g_{k_1,2}^{(1)}G^{(2)} & \cdots & g_{k_1,n_1}^{(1)}G^{(2)} \end{matrix} & & \\
& \begin{matrix} g_{11}^{(1)}G^{(2)} & g_{12}^{(1)}G^{(2)} & \cdots & g_{1,n_1}^{(1)}G^{(2)} \\ g_{21}^{(1)}G^{(2)} & g_{22}^{(1)}G^{(2)} & \cdots & g_{2,n_1}^{(1)}G^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k_1,1}^{(1)}G^{(2)} & g_{k_1,2}^{(1)}G^{(2)} & \cdots & g_{k_1,n_1}^{(1)}G^{(2)} \end{matrix} & \\
& \vdots & \ddots
\end{pmatrix}$$

$$= \begin{pmatrix}
\begin{matrix} g_{11}^{(1)} & g_{12}^{(1)} & \cdots & g_{1,n_1}^{(1)} \\ g_{21}^{(1)} & g_{22}^{(1)} & \cdots & g_{2,n_1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k_1,1}^{(1)} & g_{k_1,2}^{(1)} & \cdots & g_{k_1,n_1}^{(1)} \end{matrix} & & \\
& \begin{matrix} g_{11}^{(1)} & g_{12}^{(1)} & \cdots & g_{1,n_1}^{(1)} \\ g_{21}^{(1)} & g_{22}^{(1)} & \cdots & g_{2,n_1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k_1,1}^{(1)} & g_{k_1,2}^{(1)} & \cdots & g_{k_1,n_1}^{(1)} \end{matrix} & \\
& \vdots & \ddots
\end{pmatrix} \otimes G^{(2)}$$

Fig. 4. Tensor product decomposition of the semi-infinite band matrix derived from the generator matrix of a product code (here shown for $t = 1$).

$S^{(1)}$ is a semi-infinite band matrix with $M^{(1)} = G^{(1)}$ and overlap t. From Theorem 7 it follows that the product code is self-orthogonal if C_2 is self-orthogonal. Hence we get the following construction:

Theorem 10: Let C_1 be a classical convolutional code. Furthermore, let C_2 be a self-orthogonal code. Then the product code $C_1 \otimes C_2$ defines a quantum convolutional code, provided at least one of the following holds:

(i) Both C_1 and C_2 are linear over $GF(q)$ and C_2 is Euclidean self-orthogonal.
(ii) Both C_1 and C_2 are linear over $GF(q^2)$ and C_2 is Hermitian self-orthogonal.
(iii) C_1 is linear of $GF(p)$ and C_2 is a symplectic self-orthogonal code over $GF(p^\ell)$.

VI. Conclusion

The construction of new examples of quantum convolutional codes is a challenging task and rises several questions: what is a general framework to describe such codes, how can they be constructed, and what are the figures of merit to compare the performance of such codes? While the first of these questions has been answered in a satisfying way at least for convolutional stabilizer codes in [13], the other two questions are open (but see *e. g.* [7], [8], [12]). In this paper we have contributed to the second question by establishing a connection between product codes and convolutional codes. We have shown that the dual distance of product codes can be bounded from below which allows to obtain quantum codes for which the minimum distance is at least as large as the smaller of the minimum distances of the factors.

Concerning the third question currently not much is known, *e. g.*, the significance of notions such as *free distance* which are useful for classical convolutional codes to the quantum case has yet to be investigated.

Acknowledgment

This work was carried out while the second author was visiting IAKS. M.R. also acknowledges support by the Institute of Quantum Computing, University of Waterloo. Funding by *Deutsche Forschungsgemeinschaft (DFG), Schwerpunktprogramm Quanten-Informationsverarbeitung (SPP 1078), Projekt AQUA (Be 887/13)* is acknowledged as well.

References

[1] A. Ashikhmin and E. Knill, "Nonbinary quantum stabilizer codes," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 3065–3072, Nov. 2001, Preprint quant-ph/0005008.
[2] R. E. Blahut, *Theory and Practice of Error Control Codes*. Reading: Addison-Wesley, 1983.
[3] ——, *Algebraic Codes for Data Transmission*. Cambridge: Cambridge University Press, 2003.
[4] W. Bosma, J. J. Cannon, and C. Playoust, "The Magma Algebra System I: The User Language," *Journal of Symbolic Computation*, vol. 24, no. 3–4, pp. 235–266, 1997.
[5] A. R. Calderbank and P. W. Shor, "Good quantum error-correcting codes exist," *Physical Review A*, vol. 54, no. 2, pp. 1098–1105, Aug. 1996, preprint quant-ph/9512032.
[6] R. T. Chien and S. W. Ng, "Dual Product Codes for Correction of Multiple Low-Density Burst Errors," *IEEE Transactions on Information Theory*, vol. 19, no. 5, pp. 672–677, Sept. 1973.
[7] A. C. A. de Almeida and R. Palazzo, Jr., "A Concatenated $[(4, 1, 3)]$ Quantum Convolutional Code," in *2004 IEEE Information Theory Workshop*, San Antonio, TX, Oct. 2004.
[8] G. D. Forney, Jr. and S. Guha, "Simple rate-$1/3$ convolutional and tail-biting quantum error-correcting codes," in *Proc. ISIT'05, Adelaide, Australia*, 2005, pp. 1028–1032.
[9] M. Grassl and Th. Beth, "Cyclic quantum error-correcting codes and quantum shift registers," *Proceedings of the Royal Society London A*, vol. 456, no. 2003, pp. 2689–2706, Nov. 2000, preprint quant-ph/9910061.
[10] M. Grassl, Th. Beth, and M. Rötteler, "On Optimal Quantum Codes," *International Journal of Quantum Information*, vol. 2, no. 1, pp. 55–64, 2004, preprint quant-ph/0312164.
[11] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error–Correcting Codes*. Amsterdam: North–Holland, 1977.
[12] H. Ollivier and J.-P. Tillich, "Description of a quantum convolutional code," *Physical Review Letters*, vol. 91, 177902, Oct. 24 2003.
[13] ——, "Quantum convolutional codes: fundamentals," Nov. 2004, preprint quant-ph/0401134.
[14] A. M. Steane, "Error Correcting Codes in Quantum Theory," *Physical Review Letters*, vol. 77, no. 5, pp. 793–797, 29. July 1996.
[15] J. K. Wolf, "On Codes Derivable from the Tensor Product of Check Matrices," *IEEE Transactions on Information Theory*, vol. 11, no. 2, pp. 281–284, Apr. 1965.